Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 2, 1999

GAS FLOWS WITH HELICAL SURFACES OF THE LEVEL

S. V. Khabirov

UDC 517.944 + 533

Some properties of the invariant gas-dynamic submodel of rank 2 with spiral surfaces of the level are reported. Invariant and isobaric solutions of the submodel are considered.

1. Equations of the Submodel. The equations of gas dynamics assume a two-dimensional subalgebra which is given by the operator basis $\{\alpha t\partial_x + \partial_\theta + \alpha\partial_U, t\partial_t + (\beta t + x)\partial_x + r\partial_r + \beta\partial_U\}$, where α and β are the parameters, in a cylindrical coordinate system (t, x, r, and θ are the independent variables and U, V, and W are the velocity coordinate). This subalgebra is taken from the optimal system of subalgebras for the equations of gas dynamics with an arbitrary equation of state [1, Table 6].

The following representation of an invariant solution is chosen:

$$U = xt^{-1} + q(q^2+\alpha^2)^{-1}[q(u+\beta)+\alpha w], \quad V = v + q, \quad W = q(q^2+\alpha^2)^{-1}[qw - \alpha(u+\beta)]; \tag{1.1}$$

$$q = rt^{-1}, \qquad s = xt^{-1} - \alpha\theta - \beta\ln t, \tag{1.2}$$

where u, v, and w are the invariant velocities which are functions of the invariants q, and s, i.e., the new independent variables.

The substitution of the representation (1.1), (1.2) into the equations of gas dynamics leads to the equations of the invariant submodel

$$\begin{gathered}\rho a^2 Du + p_s = q\rho(q^2+\alpha^2)^{-2}[2\alpha q w - (q(q^2+\alpha^2) + 2\alpha^2)(u+\beta)],\\ \rho Dv + p_q = -v\rho + q\rho(q^2+\alpha^2)^{-2}(qw - \alpha u - \alpha\beta)^2,\\ Dw = -w(1+q^{-1}v), \quad A^{-1}Dp + u_s + v_q = -q^{-1}v - 3, \quad DS = 0,\end{gathered} \tag{1.3}$$

where p and ρ are the invariant pressure and density which are functions of the invariants q and s, $a^2 = q^2(q^2+\alpha^2)^{-1}$, $A = \rho c^2 = \rho f_\rho$, $p = f(\rho, S)$ is the equation of state, S is the entropy, and $D = u\partial_s + v\partial_q$. Instead of the last equation, one can use the density equation

$$D\rho + \rho(u_s + v_q + q^{-1}v + 3) = 0. \tag{1.4}$$

System (1.3) is of symmetric form, because the matrices are symmetric for the derivatives of the vector of unknown u, v, w, p, and S. If one of the matrices is positive definite, system (1.3) is symmetric and hyperbolic [2, p. 51].

Let $h(s,q) = 0$ be the equation of i-characteristics. There is a three-multiple contact i-characteristic C_0: $uh_s + vh_q = 0$. The other two possible real i-characteristics satisfy the equation

$$C_\pm: \quad (a^2u^2 - c^2)h_s^2 + 2a^2uvh_sh_q + a^2(v^2-c^2)h_q^2 = 0. \tag{1.5}$$

The hyperbolicity condition of system (1.3) is the nonnegativity condition for the discriminant of the square equation (1.5) relative to $h_sh_q^{-1}$, and sets the i-domain of hyperbolicity on this i-solution:

$$a^2u^2 + v^2 \geqslant c^2. \tag{1.6}$$

Institute of Mechanics, Ufa Scientific Center, Russian Academy of Sciences, Ufa 450000. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 2, pp. 34–39, March–April, 1999. Original article submitted June 3, 1998.

TABLE 1

A	α, β	Operators
$pg(p\rho^{-1})$	$\alpha^2+\beta^2 \neq 0$	X_p
$g(\rho)$		X_1
ρ		X_1, X_p
0		$X_{\chi(p)}$
$pg(p\rho^{-\gamma}), \gamma \neq 1$	$\alpha=\beta=0$	Y_γ
$pg(\rho)$		Y_0+2X_p
γp		X_p, Y_0
$g(\rho e^p)$		Y_0+2X_1
$\rho^\gamma, \gamma \neq 1$		Y_γ, X_1
0		$Y_0, X_{\chi(p)}$

In physical variables, condition (1.6) takes the form

$$(V-rt^{-1})^2+r^2(r^2+\alpha^2t^2)^{-1}(U-\alpha tr^{-1}W-xt^{-1}-\beta)^2 \geqslant c^2.$$

As $r \to 0$, this condition becomes the supersonic-flow condition for a projection onto the plane perpendicular to the x axis.

For the invariant submodel (1.3), (1.4), one can introduce quantities and definitions similar to those of two-dimensional steady-state flows: the i-streamline $u^{-1}ds = v^{-1}dq$, which is the bicharacteristic of the i-characteristic C_0, the stream i-function $\psi(s,q)$ for which $u=-(3/2)q\rho\psi_q$, $v=(3/2)q\rho\psi_s$, $\rho=\varphi_\psi\varphi_s^{-1}$, and $q=\varphi_s(s,\psi)$, the i-integral of entropy $S=S(\psi)$, and the i-integral of twisting $w=D(\psi)\varphi_\psi^{2/3}\varphi_s^{-1}$.

2. Group Property. System (1.3), where (1.4) is used instead of the last equation, admits the following equivalence transformations: $\rho'=a_1\rho$, $p'=a_1p+a_2$, and $A'=a_1A$ for $\alpha \neq 0$, and $(s',q',u',v',w',\beta')=a_3(s,q,u,v,w,\beta)$ and $p'=a_3^2p$ for $\alpha=0$; the transformation of the invariant variables is omitted. The group classification of the submodel equations with respect to arbitrary elements A, α, and β is performed within the equivalence transformations. The result is given in Table 1, where the following notation is introduced: $Y_\gamma = s\partial_s + q\partial_q + u\partial_u + v\partial_v + w\partial_w - 2\rho\partial_\rho + 2\gamma(\gamma-1)^{-1}p\partial_p$ and $X_\chi(p)=\chi'\rho\partial_\rho+\chi\partial_p$ (χ and g are arbitrary functions). The kernel of the admitted algebras is one-dimensional $\{\partial_s\}$.

3. Level Surfaces of the Invariant Functions and the Univalence Domain. The two-dimensional surface in the physical space $R^4(x,r,\theta,t)$ corresponds to the point (s,q) of the invariant submodel (1.3). A section by the hyperplane $t=\text{const}$ gives a curve in $R^3(x,r,\theta)$. For various values of t, the projections of the curves onto the space $R^3(x,r,\theta)$ form a surface which is a helical surface of the level of invariant functions the values of which are calculated at one point. The equation of the level surface is derived from formulas (1.2) after the elimination of t:

$$x = rq^{-1}(\alpha\theta+s-\beta\ln q+\beta\ln r). \tag{3.1}$$

A section of the surface (3.1) by the half-plane $\theta=\text{const}$ is the curve having two zeros of the function $x(r)$ at the points $r_0=0$ and $r_1=q\exp(-\beta^{-1}(s+\alpha\theta))$ and one minimum $er_m=r_1$, $ex_m=-\beta r_1$. As the angle θ increases, the quantities r_1 and $|x_m|$ decrease (see Fig. 1).

A section of the surface (3.1) by the plane $x=x_0$ is the helix for $x_0 \geqslant 0$ or two helices with the common point $\alpha\theta_0+s=\beta(\ln\beta-1-\ln|x_0|)$, $\beta r_0=-qx_0$ according to the equation $\alpha\theta = x_0qr^{-1}-s+\beta\ln q-\beta\ln r$ for $x_0<0$.

For $\alpha=0$, the level surfaces are the cylindrical surfaces formed by rotation of the curve (3.1) about the x axis.

Let $\omega \subset R^2(s,q)$ be the domain in the half-plane $q>0$. For fixed t, the lines in $R^3(x,r,\theta)$ which cover the domain Ω correspond to the points of the domain ω. Let Ω_{θ_0} be the cut of the domain by the half-plane $\theta=\theta_0$. There is a one-to-one correspondence between ω and Ω_{θ_0} according to formulas (1.2). With variation in θ_0 on $2\pi k$, i.e., for $\theta=\theta_0+2\pi k$ (k is an integer), the image Ω_{θ_0} is shifted along the x axis by $2\pi k\alpha t$ and both images are in the same half-plane. For the images to be univalent, it is necessary and sufficient that the

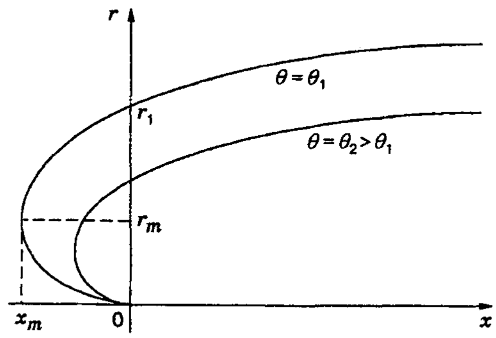

Fig. 1

width of ω along s, i.e., the length of the cut of ω by the straight line $q = q_0$, be not greater than $2\pi\alpha$. For example, one can take ω from the half-band $\{q \geqslant 0, |s| \leqslant \pi\alpha\}$.

If the width of ω along s is equal to $2\pi\alpha$, for a certain value of $q = q_0$, the discontinuities of the physical quantities U, V W, ρ, and p can occur on the helix $s = s_0$, $q = q_0$ in the physical space for fixed t. For continuity, the periodicity of the invariant functions of s with period $2\pi\alpha$ is required.

Thus, a continuous invariant flow in the whole space can occur if there is a periodic solution of submodel (1.3) in a domain of width $2\pi\alpha$ with respect to s for $q \geqslant 0$.

If the values of the invariant functions at the points $q = q_0$, $s_1 = s_0$, and $s_2 = s_0 + 2\pi\alpha$ are different, what should they be for the helical discontinuity surface to become a contact discontinuity, a shock wave, or a wall?

Proposition 1. *The physical trajectories lie on the surfaces which correspond to the i-streamlines of the submodel.*

Proof. The equations for calculation of trajectories by formulas (1.1) have the form

$$\frac{dx}{dt} = \frac{x}{t} + \frac{q^2(u+\beta) + \alpha q w}{q^2+\alpha^2}, \quad \frac{dr}{dt} = v + q, \quad r\frac{d\theta}{dt} = \frac{q^2 w - \alpha q(u+\beta)}{q^2+\alpha^2}. \tag{3.2}$$

The equalities $dq/dt = v/t$ and $ds/dt = u/t$ follow from (1.2) and (3.2). As a result, the equation for the i-streamline $u^{-1}ds = v^{-1}dq$ is derived.

Consequence. If the domain of definition ω of the solution of submodel (1.3) is bounded by the i-streamlines and its width relative to s does not exceed $2\pi\alpha$, a moving wall or contact discontinuity corresponds to the boundary of ω in the physical space. In particular, a flow in which a helical wall of zero thickness moves corresponds to a curvilinear semi-band $q \geqslant 0$ of width $2\pi\alpha$ relative to s bounded by the i-streamlines. If the pressure is continuous on the surface, a contact discontinuity occurs.

4. Strong-Discontinuity Equations. The invariant surface is given by the equality

$$F(x,r,\theta,t) = q - h(s) = 0. \tag{4.1}$$

The normal in the physical space and the motion velocity of the surface in the direction of the normal are calculated by the formulas

$$\boldsymbol{n} = |\nabla F|^{-1}\nabla F = (1 + (1+\alpha^2 q^{-2})h_s^2)^{-1/2}(-h_s, 1, \alpha q^{-1}h_s),$$
$$D = -|\nabla F|^{-1}F_t = (q - (\beta + xt^{-1})h_s)(1 + (1+\alpha^2 q^{-2})h_s^2)^{-1/2},$$

where $\nabla = (\partial_x, \partial_r, r^{-1}\partial_\theta)$.

The velocity vector is decomposed into the normal and tangent components, $\boldsymbol{u} = (U, V, W) = u_n\boldsymbol{n} + \boldsymbol{u}_\sigma$ and $u_n = \boldsymbol{u}\cdot\boldsymbol{n}$.

The relative velocity and the conditions on the strong-discontinuity surface [2, p. 39] are written via the invariants:

$$\omega = u_n - D = (v - uh_s)(1 + (1+\alpha^2 q^{-2})h_s^2)^{-1/2}. \tag{4.2}$$

For contact discontinuity, we have $[p] = p_2 - p_1 = 0$ and $v_j - u_j h_s = 0$, where the subscript $j = 1$ and 2 determines the value of the parameters on both sides of the discontinuity.

For the shock wave,

$$\omega_1^2 = \rho_2 \rho_1^{-1} [p][\rho]^{-1}, \quad \omega_2^2 = \rho_1 \rho_2^{-1} [p][\rho]^{-1}, \tag{4.3}$$

the Hugoniot condition has the form $H(\rho_2, p_2; \rho_1, p_1) = \varepsilon_2 - \varepsilon_1 - (1/2)(p_2 + p_1)(\rho_1^{-1} - \rho_2^{-1}) = 0$, where $\varepsilon_j = \varepsilon(\rho_j, p_j)$ is the value of the internal energy from the ith side of the shock wave, and $[\boldsymbol{u}_\sigma] = 0$.

The following alternative follows from the last condition:

1) the i-direct shock wave ($h_s = 0$)

$$[u] = [w] = 0, \qquad [v] = [\omega]; \tag{4.4}$$

2) the i-oblique shock wave ($h_s \neq 0$)

$$[u] = -(\alpha^2 h^{-2} + 1) h_s [v], \qquad [w] = 0, \qquad [\omega] = [v](1 + (1 + \alpha^2 h^{-2}) h_s^2)^{1/2}. \tag{4.5}$$

Proposition 2. *The other flow parameters are determined from the impact-transition conditions with the use of the specified flow parameters in front of the shock wave under the hyperbolicity condition* (1.6) *in the case of an i-oblique shock wave and one flow parameter behind the shock-wave front* ρ_2, p_2, and $h(s)$.

Proof. Let ρ_1, p_1, u_1, v_1, and w_1 be given and, for example, $\rho_2 > \rho_1$. Then, p_2 is determined from the Hugoniot condition. The quantities ω_1 and ω_2 are calculated from (4.3). In the case of an i-direct shock wave, the quantities u_2, w_2, and v_2 are found from (4.4).

For a i-oblique shock wave, the equation of its surface (4.1) is determined from (4.2) from the side of the specified parameters: $(u_1^2 - \omega_1^2(1 + \alpha^2 h^{-2})) h_s^2 - 2 v_1 u_1 h_s + v_1^2 - \omega_1^2 = 0$.

Real solutions are possible under the condition that the discriminant of this square equation is nonnegative relative to h_s:

$$v_1^2 + h^2 (h^2 + \alpha^2)^{-1} u_1^2 \geqslant \omega_1^2 = \rho_2 \rho_1 [p][\rho]^{-1} \geqslant c_1^2.$$

This inequality coincides with (1.6) from the side of the specified parameters. After $h(s)$ has been determined, the quantities w_2, v_2, and u_2 are determined from (4.5).

If p_2 is given instead of ρ_2, then ρ_2 is determined from the Hugoniot condition.

After $h(s)$ is given, ω_1 is calculated from (4.3), and the Hugoniot condition and the first equality in (4.5) set ρ_2 and p_2 in the case of a normal gas [2, pp. 23 and 50].

If v_2 is given, i.e., $[v]$, for an i-oblique shock wave, from (4.2), (4.3), and (4.5) we determine $p_2 = p_1 - \rho_1 [v](v_1 - u_1 h_s)$ and $\rho_2^{-1} = \rho_1^{-1}(1 + [v](v_1 - u_1 h_s)^{-1}(1 + (1 + \alpha^2 h^{-2}) h_s^2))$. The substitution of these expressions into the Hugoniot condition gives a differential equation for the determination of $h(s)$. If u_2 is specified, the i-oblique shock wave is resolved similarly.

5. Invariant Solution. The invariant ∂_s-solution is constructed on the kernel of algebras admitted by the submodel. This is the s-independent solution of system (1.3). Since $v \neq 0$, the flow is isentropic: $S = S_0$. The integral

$$D^3 \rho v = q^2 w^3 \tag{5.1}$$

holds. After the replacement $\lambda = qw - \alpha(u + \beta)$ and the substitution of the integrals we obtain the following system of ordinary differential equations:

$$\begin{gathered} \frac{\lambda_q}{\lambda} + \frac{1}{v}\Big(1 + \frac{2\alpha^2}{q(q^2 + \alpha^2)}\Big) = 0, \\ v v_q + v + \rho^{-1} c^2(\rho) \rho_q = q(q^2 + \alpha^2)^{-2} \lambda^2, \quad \frac{\rho_q}{\rho} + \frac{v_q}{v} + \frac{1}{q} + \frac{3}{v} = 0. \\ v v_q + v + \rho^{-1} c^2(\rho) \rho_q = q(q^2 + \alpha^2)^{-2} \lambda^2, \quad \frac{\rho_q}{\rho} + \frac{v_q}{v} + \frac{1}{q} + \frac{3}{v} = 0. \end{gathered} \tag{5.2}$$

For $\alpha = 0$, there is one more integral $u+\beta = Cqw$. There is no Bernoulli-type integral in the submodel considered.

The equation of state of the gas can be chosen in such a way that the specified velocity v is obtained. For example, for $v = -q$, from (5.1) and (5.2), it follows that $\rho = \rho_0 q$, $w = -D\rho_0^{1/3}$, $u = -\beta - \alpha^{-1}D\rho_0^{1/3} + (\alpha^{-1}q + \alpha q^{-1})c(\rho_0 q)$, and $c(\rho) = c_0\rho^2(\rho^2 + \alpha^2\rho_0^2)^{-1}\exp[-2\rho_0\rho^{-1} - 2\alpha^{-1}\arctan(\rho\alpha^{-1}\rho_0^{-1})]$. This solution is determined for $q > 0$ and describes the flow along the x axis with spreading-free twisting. For $q = 0$, vacuum occurs.

6. Isobaric Flows. A general solution of the equations of gas dynamics at constant pressure $p = p_0$ was found by Ovsyannikov [3]. For invariant submodels, the compatibility should be studied. The general solution is written in the following Lagrangian variables: $r = R$, $\theta = \vartheta$, and $x = \xi$ for $t = 1$.

It is convenient to use the polar coordinates related to the cylindrical formulas: $V = Q\cos\varphi$, $W = Q\sin\varphi$, and $\varphi = \psi - \theta$. The equations of isobaric flows take the form

$$U_\xi + (Q_R - R^{-1}Q\psi_\vartheta)\cos\varphi + (R^{-1}Q_\vartheta - Q\psi_R)\sin\varphi = 0,$$

$$V_\xi^2 - R^{-1}Q(Q_R\psi_\vartheta - \psi_R Q_\vartheta) = (QU_R\psi_\xi - R^{-1}U_\vartheta Q_\xi)\sin\varphi - (U_R Q_\xi - R^{-1}QU_\vartheta\psi_\xi)\cos\varphi,$$

$$|M| = \frac{\partial(U, Q, \psi)}{\partial(\xi, R, \vartheta)} = 0.$$

Isobaric solutions are classified with respect to the rank of the matrix M [3]. The constant solution

$$U = u_0, \quad V = v_0\cos\vartheta + w\sin\vartheta \quad W = -v_0\sin\vartheta + w_0\cos\vartheta$$

corresponds to the zero rank.

The representation of the solutions in the form

$$F(p, R\cos(\mu - \psi + \vartheta), \xi(Q'^2 + Q^2\psi'^2)^{1/2} - RU'\sin(\mu - \psi + \vartheta)) = 0,$$

where $\tan\mu = Q'(Q\psi')^{-1}$ and ψ, Q, and U are arbitrary functions of the parameter $p(\xi, R, \vartheta)$, and F is an arbitrary function, corresponds to rank 1.

The representation

$$f(U, Q, \psi) = 0, \qquad \xi f_U + R\cos(\psi - \vartheta)F_Q - RQ^{-1}\sin(\psi - \vartheta)F_\psi = g(U, Q, \psi),$$

$$\xi^2 f_{UU} + R^2\cos^2(\psi - \vartheta)F_{QQ} + R^2Q^{-2}\sin^2(\psi - \vartheta)f_{\psi\psi} + 2\xi R\cos(\psi - \vartheta)f_{UQ}$$

$$-2\xi RQ^{-1}\sin(\psi - \vartheta)f_{U\psi} - R^2Q^{-1}\sin 2(\psi - \vartheta)f_{Q\psi} + R^2Q^{-1}\sin^2(\psi - \vartheta)f_Q$$

$$+R^2Q^{-2}\sin 2(\psi - \vartheta)f_\psi - 2\xi g_U - 2R\cos(\psi - \vartheta)g_Q + 2RQ^{-1}\sin(\psi - \vartheta)g_\psi = h(U, Q, \vartheta),$$

where f, g, and h are arbitrary functions, corresponds to rank 2 of the matrix M.

For the invariant solution (1.1), the formulas

$$Q^2 = (v + R)^2 + R^2(R^2 + \alpha^2)^{-2}(Rw - \alpha(u + \beta))^2,$$

$$\psi = \vartheta + \varphi, \qquad \tan\varphi = R(Rw - \alpha(u + \beta))(R^2 + \alpha^2)^{-1}(v + R)^{-1}, \qquad \xi = s + \alpha\vartheta$$

hold.

The invariant solutions with rank 2 of the matrix M have the form

$$U = \alpha\psi + f(Q), \qquad s - \alpha\varphi + \alpha RQ^{-1}\sin\varphi - Rf'\cos\varphi = g(Q),$$

$$-f''R^2\cos^2\varphi + 2\alpha RQ^{-1}(1 - R\cos\varphi)\sin\varphi - R^2Q^{-1}f'\sin^2\varphi - 2Rg'\cos\varphi = h(Q)$$

with three arbitrary functions f, g, and h.

There are no constant invariant solutions.

There are two invariant solutions with rank 1 of the matrix M only for $\beta = 0$:

$$\alpha \neq 0, \quad u = -s, \quad v = -q, \quad w = -\alpha q^{-1}s;$$

$$\alpha = 0, \quad u = -s, \quad v = -q + Bq^{-1}(B + Cs), \quad w = Bq^{-1}(q^2 - (B + Cs)^2)^{1/2},$$

where B and C are the constants.

The centered wave $U = xt^{-1}$, $V = W = 0$ corresponds to the first solution in physical variables. In the second case, in physical variables the solution $U = 0$, $V^2 + W^2 = B^2$, $V = Br^{-1}(Bt + Cx)$ is obtained.

The particle trajectories of the particles have the following form:

$$x = \xi, \quad r^2 = R^2 + B(t-1)(B(t+1) + 2C\xi),$$

$$\tan(\theta - \vartheta) = B(t-1)(R^2 - (C\xi + B)^2)^{1/2}(R^2 + B(t-1)(C\xi + B))^{-1}.$$

The particle trajectories $x = \xi$, $r\cos(\theta - \vartheta - \arctan(b^2 - 1)^{-1/2}) = (C\xi + B)(b^2 - 1)^{1/2}$, and $b = R(C\xi + B)^{-1}$ are the half lines tangent to the circle $r = (C\xi + B)(b^2 - 1)^{1/2}$ and represent the gas outflow from a volumetric source with constant velocity modulus and twisting.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 96-01-01780).

REFERENCES

1. L. V. Ovsyannikov, "Program SUBMODEL. Gas dynamics," *Prikl. Mat. Mekh.*, **58**, No. 4, 30–55 (1994).
2. L. V. Ovsyannikov, *Lectures on Gas Dynamics* [in Russian], Nauka, Moscow (1981).
3. L. V. Ovsyannikov, "Isobaric motions of gas," *Differ. Uravn.*, **30**, No. 10, 1792–1799 (1994).